Our Patch-work System.—In most schools we begin our mathematical education by leaving great gaps in the subject matter. For instance, algebra and geometry are developed as distinct subjects and the gap between them is gradually filled up as we proceed. If it is found feasible to start our mathematical farm by the cultivation of isolated patches, it may possibly be found feasible to have isolated patches in our methods of work and in our centers of interest. It may be that at the time the student reaches the university he ought to get into a new world with new views, new objects, new perspectives, and new mathematical ideals.

We are sometimes told that our first year of university instruction is bad because those who stood the highest in their classes in the high school do not make as good freshman records as some of those who did only mediocre work in the high school. Taken by itself this does not prove anything. The mediocre student in the high school may appear mediocre only in view of some arbitrary and unjust standards, or it may be that these standards are just at the high school age but would not be just later in life. The qualities leading to distinction in youth do not always lead to equal distinction in manhood. Adaptation is one of the most important faculties to cultivate, and if the student learns, at the beginning of his college course, the hard lesson that he has sometimes to adapt himself to new and radically different conditions, he has learned one of the most important lessons of life. For instance, young people do not grow gradually into the married state but they are required to adapt themselves at one step. Discontinuity is common in life and we may naturally expect to meet it in education.

Inherent Difference.—Before trying to determine whether the extensive freshman fatalities are due to high school or to college methods of instruction let us agree that these two methods ought to be different. After agreeing on this point we still have the main difficulty to solve; namely, how great this difference should be. On this point, we can probably not expect to reach perfect agreement. The more advanced the work is the more the individuality of the teacher does and ought to assert itself. Hence we should not expect as much uniformity among the methods used by freshman instructors as among those employed by the high school teachers. At this point we reach again the perplexing question how great this difference ought to be.

While my arguments thus far have been directed mainly towards placing the freshman situation in a charitable light, I would not desire to be classed with those who are entirely satisfied with the situation and who see no room for improvement. Without trying to turn high school teachers into university teachers or vice versa, I would urge most strongly that these two classes of teachers should regard it as one of their most important duties to learn from each other. The tendency to grow away from each other is exceedingly strong, and a conscious and repeated effort to overcome this tendency is necessary, in most cases, in order to maintain ourselves in a condition to render the best service. There is much that we can learn from each other.

Higher Ideals Needed.—The high school teachers, as a class, need higher scholarly ideals. The high school teachers' meetings should have more papers dealing with modern mathematical subjects, and the high school teachers' library should contain more books and journals dealing with advanced mathematics. A teacher who has ceased to hunger for more knowledge about his subject has no business in the high school. Let us not deceive ourselves. Our students know where our interests lie, and how can we expect to inspire them with zeal in the subject matter if we have not sufficient interest in it to work ahead to the best of our abilities and in accordance with our opportunities.

Mathematical Journals.—One of the chief incentives to come to a meeting of this kind should be the opportunity it gives us to learn, by consultation with each other or by means of the excellent library at this place, more about some mathematical questions. The mathematical journals should be one of the chief centers of interest, since they are of such varied grades as to come within the reach of all. At this time high school teachers should have especial interest in the improvement of the AMERICAN MATHEMATICAL MONTHLY, which is peculiarly their own journal in this country, from the present point of view. The *Mathematical Gazette* of England has somewhat similar aims.

Those who read foreign languages and who desire a journal of about the same grade in such a language, would probably find the following (in French, German, Italian, and Spanish, respectively) very instructive: *L'Enseignement Mathématique, Zeitschrift für Mathematischen und Naturwissenschaftlichen Unterricht, Periodico di Matematica,* and *Revista de la Sociedad Matemática Española.* Some of the other foreign languages which are less extensively read by Americans have secondary mathematical journals. For instance, Japan has such a journal which is known as "The XY" and is published in the Japanese language by the XY House. A brief review of this journal is found in the *Mathematical Gazette* for January, 1913, page 17.

Interest in Pedagogical Questions.—While the high school teacher can do much towards decreasing the gap at the beginning of the freshman year by taking a deeper interest in the subject matter and in the fields which lie just beyond those with which he is primarily concerned, the university instructor of freshman mathematics should not forget that the strong temptations to grow away from pedagogic interest must be partially counteracted. Some of our European colleagues are setting us conspicuous examples along this line. The fact that the office of President of the Mathematical Association of England, an association of teachers and students of elementary mathematics, has been filled in recent years by such eminent men as A. R. Forsyth, B. G. Mathews, G. H. Bryan, H. H. Turner, E. W. Hobson and Sir George Greenhill, illustrates the attitude of some of the most prominent English mathematicians with respect to the work of the secondary teachers.

A few of the leading American mathematicians are also taking a deep interest in high school mathematics, but I do not know of any American association of teachers of elementary or secondary mathematics which can point to a list of

such eminent successive presidents as the one which has just been cited. Whether this is due to a lack of appreciation of scientific eminence on the part of our teachers of secondary mathematics, or whether it is mainly due to the fact that too few eminent American mathematicians are interested in elementary and in secondary mathematics, may be difficult to determine. Probably both of these undesirable conditions have their influence and tend to strengthen each other. At any rate, our leading mathematicians and our teachers of secondary mathematics should be in closer touch with each other.

Great Learning and Simple Language.—The teachers of elementary mathematics are sometimes afraid of the university teachers because the latter use a language of their own. This is too often true, but it is not true of all the really great men. Great *learning* tends towards a simple language while great *cramming* tends toward an unintelligible jargon. If a man cannot talk so as to be understood you can be almost sure that he has read much but understood little, or that he has dreamed himself into many problems but did not take the time to solve any of them completely. Such dreamers may be helpful to the research student but they have little for the busy teacher.

On the other hand, the great mathematician who has really mastered many of the problems of advanced mathematics should be able to talk most clearly about the difficult mathematical questions which confront the teacher of secondary mathematics. Hence these teachers should seek great mathematicians as presidents of their associations in order that they may be able to learn new things easily at the meetings, and thus be enabled to present subjects more clearly to their students.

Among the subjects which are at the present time of common interest to many advanced mathematicians and to high school teachers, the foundations of geometry and the theories of aggregates and groups deserve especial mention.

Special Class Preparation Needed.—The instructor of the freshmen students of our colleges probably fails more frequently on account of a lack of a complete mastery of the subject which he is to teach than on account of the fact that his methods differ from those of the high school. The student expects and welcomes a change of method. He likes to get into a new atmosphere. He welcomes the greater freedom and the greater individuality of methods which he finds in college. The great disturbing element often is a lack of clearness and simplicity. This may be due to the fact that his instructor was thinking along an entirely different line until five minutes before the class. What we need most of all is a deeper interest in the subject matter on the part of our teachers, and when our teachers' conferences come to be dominated more completely by this thought, our university instructors and high school teachers will have more in common, and the gap at the beginning of the university course will become the freshman's delight instead of his stumbling block.

Resumé.—The substance of our argument is, in brief, as follows: The scientific interests of the high school teacher and the university instructor should have more in common. As the latter have generally better opportunities and more

time for study, the former should more frequently invite the latter to assist them in the solution of their more difficult scientific questions. On the other hand, the university instructor should realize that his freshman classes deal with subjects which he himself understands imperfectly, and he should be interested in arriving continually at clearer views in regard to these subjects and the methods of presenting them.

The Investigator and the Teacher.—In recent years there has been considerable discussion in reference to making a public distinction between the investigator and the teacher on the university faculty. Full professors of mathematics have been appointed, in excellent universities, mainly on the ground of their success as teachers. The question might be raised whether it would not be wiser to insist that every successful university instructor must be an investigator, but at the same time to admit that investigation in *elementary* subjects is also important, and especially in respect to the teaching of these subjects. Those who are investigating in the fields of freshman and high school mathematics should, and in many cases now actually do, fill the so-called teaching professorships.

Mathematics a Growing Subject.—One of the most harmful phrases is the assertion that "mathematics is a finished science." It would be difficult to find a shorter way to expose dense ignorance of the subject than to use this phrase seriously. Even elementary mathematics is not a finished science and indeed no one has a complete knowledge of any elementary subject.

The feeling that we are all dealing with a growing subject and that new developments in one branch are apt to throw light on other branches should awaken in us a hopeful interest in the work in the different branches. Such an interest tends to inspire us with a zeal that is contagious. This zeal is one of the greatest assets of any teacher of mathematics. It serves also as a bond which unites us all and increases our respect for the work in the various fields of mathematics. The university instructor who has great respect for his high school colleague is almost sure to be a good teacher, and the high school teacher who honors the work which his university colleague tries to do is apt to continue to grow and to inspire his students with his own zeal.

The Beauties of Mathematics.—We are all inspired by the beauties of this country when viewed from the tops of the various hills and mountains of this state. The views from this campus enthuse us and lead us to express our delight to those around us. The descriptions of the beauty of the "Golden Gate" have aroused most interesting impressions in the minds of millions of school children, as well as in the minds of adults who have never visited this part of the world. Is it not a shame that the intellectual beauties seen from intellectual heights should be passed by without also awakening our emotions? Elementary algebra and elementary geometry are largely blooming orchards. The fruit will appear later. Let us not talk continually about the fruit now. The tendency of the modern text-book is to disfigure a beautiful blooming tree by tying pictures of the fruit on its branches. I believe that more students can be inspired by the blossoms which they see than by the pictures of the fruit. At any rate, let us be

alive to the beauties of our subjects, no matter how elementary. In this way most freshman mathematical troubles will vanish notwithstanding the change of emphasis.

Let me carry this figure a little further. What I mean is that the high school teacher should have traveled widely through the beauties of mathematics and should not hesitate to talk about some of the beauties which are beyond his students. We do not hesitate to talk about the Grand Canyon of the Colorado or about the grandeur of Yosemite to those who never expect to visit these places. Why should we then hesitate to mention some of the jewels of the calculus, the grandeur of number realms, or the wonders of higher dimensions, even to those who are not sure that they will ever reach these stages. In most cases the answer is evident. We know too little ourselves about matters lying close to the subjects we teach, and therefore we cannot talk about these things in a sufficiently simple language. We have not seen the subjects ourselves but merely read about them. Teaching the most advanced subject which we have studied is like describing a country about which we have read but which we have not seen.

The Chief Essential of Good Teaching.—If I were asked to provide a formula for the manufacture of successful high school and university teachers, I should consider it necessary to include a great number of ingredients. Hence my emphasis on more scholarship *relating directly to the subject in hand* means simply that from my point of view the supply of this ingredient is, at the present time, more commonly inadequate than that of any other. If we could add a sufficiently large amount of this, together with the zeal which naturally goes with it, to the present equipment of our teachers, most of the present freshman troubles would vanish. Possibly the great prominence of athletic matters as chief subjects of conversation might even become a thing of the past.

Journals and Books.—While it is expensive to study at a university or to attend a large number of meetings, it is comparatively inexpensive to secure a few good journals and good books. Most of the best thoughts of the world find their way into journals and then into books,[1] and the most economical way of thinking is to relate ourselves to these fresh thoughts. Hence the need of good libraries is being recognized as never before, and the teacher's private library should be as complete as possible along the lines in which his work lies. As his subject connects with other realms of growing knowledge, such a library will enable him to relate his life with the sympathies and interest of all seekers of truth. This vital connection ennobles our lives as teachers and furnishes the most important element of that inspiration and sympathy between teacher and student which surmounts all obstacles.

Journals as Antidotes.—The rapid increase in the number and types of mathematical journals during the last fifty years reflects their great value. While Americans are generally quick to adopt the best, the American teachers of mathematics have been very slow in improving their efficiency by reading the mathe-

[1] Cajori's history of logarithms which has been appearing in the MONTHLY is now in the journal stage.

matical journals. The reading of the same things goes far toward creating points of contact and developing common interests, and it is these common interests between high school and university instructors which will tend to remove the objectionable features of the gap with which the freshman has to contend. When an evil in the physical world begins to assume serious proportions, its antidotes become prominent objects of study. One of the strongest antidotes against the forces which tend to produce harmful gaps in our mathematical education may be found in the uniting tendencies of the mathematical journals. As these journals constitute such a strong directing and inspiring force and provide a forum where all may meet at will for the discussion of problems, they should be heartily supported by all teachers of mathematics.

In closing I desire to lay great stress on another point, namely the history of the subjects we teach. The normal student is most deeply interested in some historical facts but most of us are too lazy or too indifferent to study the history of our subject sufficiently to speak about it in simple and accurate language. A few historical observations now and then put new life into our classes. It is all the more necessary that we cultivate a continuous acquaintance with the history of our subjects since the recent progress of our knowledge along this line has been so rapid. In history we find a bond which unites not only all those interested in mathematics but also all who are interested in the development of the intellect.

ON THE SOLUTIONS OF LINEAR EQUATIONS HAVING SMALL DETERMINANTS.

By F. R. MOULTON, University of Chicago.

The fundamental theorem respecting linear non-homogenous equations is that their solution exists and is unique if the determinant of the coefficients of the undetermined quantities is distinct from zero. But this simple theorem does not by any means contain all that is important from a practical point of view. If the coefficients of the equations are furnished by observations, or by measurements, they are determined only to a certain number of places. The question then arises how exact their solution is. In the discussion of this question the magnitude of the determinant plays an important rôle, and the results are particularly interesting and striking when the determinant is small.

For the sake of illustration of some of the peculiarities of the solutions of linear systems, before taking up the general discussion, consider the equations

$$(1) \qquad \begin{cases} .34622\,x + .35381\,y + .36518\,z = .24561, \\ .89318\,x + .90274\,y + .91143\,z = .62433, \\ .22431\,x + .23642\,y + .24375\,z = .17145. \end{cases}$$

It will be supposed that the coefficients and right members of these equations are furnished by observations, and that the possible errors to which they are subject do not exceed five units in the sixth place. The problem is to determine x, y, and z and to find the degree of accuracy of the determination.

A system of linear equations turns up in many problems of applied mathematics. Indeed, it was such systems in the theory of the determination of the orbits of the heavenly bodies from observations from the earth which led to the considerations which are developed here. They arise in all problems involving the determination of quantities by the method of least squares. In these computations it is of course essential that the work be done with numerical precision, and this has led to systematic modes of procedure and the development of control formulas for use in the process. The method, aside from the control formulas, for solving linear systems is to eliminate one unknown from all of the equations except one by use of the one. Then, one equation of the reduced set is used to eliminate a second unknown. The process is repeated until a single equation remains involving a single unknown.

In applying the process for solving linear equations, which has been described, to equations (1) no difficulty is encountered. It is true that a zero turns up at the successive stages in the first place on the left, but it is natural to multiply such equations by 10, or to add another figure on the right. It is found on carrying out the solution with seven-place tables that

$$x = -1.027066, \quad y = +2.091962, \quad z = -0.380515. \tag{2}$$

On substituting the values of x, y, and z in (1) it is found that these equations are satisfied to the last place. Moreover, if either x, y, or z alone is changed by so much as one unit in the fifth place the equations will no longer be satisfied. But

$$\begin{cases} x = -1.022773, & y = +2.084125, & z = -0.376941, \\ x = -1.031229, & y = +2.099457, & z = -0.383879, \end{cases} \tag{3}$$

also satisfy equations (1) up to the last place. The variation in x is nearly one per cent., and in z nearly two per cent. That is, although equations (1) are supposed to be accurate to five places, their solution is determinate only to two places; or, expressed otherwise, the uncertainty in the result is a thousand times what might be expected.

The phenomenon exhibited by (1), (2), and (3) is to be explained. If equations (1) are written in the form

$$\begin{cases} a_1x + b_1y + c_1z = d_1, \\ a_2x + b_2y + c_2z = d_2, \\ a_3x + b_3y + c_3z = d_3, \end{cases} \tag{4}$$

the expression for x is

$$x = \frac{\begin{vmatrix} d_1, & b_1, & c_1 \\ d_2, & b_2, & c_2 \\ d_3, & b_3, & c_3 \end{vmatrix}}{\begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix}} = \frac{D_x}{D}. \tag{5}$$

There are similar expressions for y and z.

Suppose the numerical value of D is small. If a_i, b_i, and c_i are defined only to five places and the first three places in D are zeros, D is in reality given only to two places. Consequently x is defined only to two places. In the numerical example D is small, and this is the explanation of the variations in the solutions. The value of D is

$$D = \begin{vmatrix} .34622, & .35281 & .36518 \\ .89318, & .90274 & .91143 \\ .22431, & .23642 & .24375 \end{vmatrix}.$$

Let the second column be subtracted from the first and third columns, and let the resulting third column be added to the first. These operations do not change the value of the determinant which then takes the form

$$D = \begin{vmatrix} +\ .00378 & .35381 & .01137 \\ -\ .00087 & .90274 & .00869 \\ -\ .00478 & .23642 & .00733 \end{vmatrix},$$

which is necessarily small because the elements of both the first and the third columns are small.

Suppose the coefficients a_i, b_i, c_i, and d_i are given by observations to a certain number of places, and consider the problem of finding the greatest variations possible in x, y, and z. Suppose the determinant D is not zero, though it may be small; then equations (4), regarding the a_i, b_i, c_i, and d_i as being exact, have a unique solution. The assumption that the coefficients are exact is equivalent to saying that all the digits beyond the number of places given are zero (assuming that all numbers are given in the decimal form). Let x, y, and z represent the solution of (4). In order to use effectively the hypothesis that the digits beyond those given are zero it will be necessary to use more places in the computations than are given in the coefficients. This was done in computing (2), and when these values of x, y, and z are substituted in (1) the digits in the sixth places are zeros.

Suppose the exact values of the coefficients are

$$(6) \qquad a_i + \alpha_i, \quad b_i + \beta_i, \quad c_i + \gamma_i, \quad d_i + \delta_i \quad (i = 1, 2, 3),$$

where a_i, b_i, c_i, and d_i are the numbers furnished by the observations, and where α_i, β_i, γ_i, and δ_i are the differences between these numbers and the true values of the quantities measured. The latter quantities are, of course, not known, but generally limits can be placed on their possible magnitudes. For example, it might be that they would not exceed five units in the first neglected place. The problem is to find the greatest admissible variations in x, y, and z for all admissible variations in the α_i, β_i, γ_i, and δ_i. For any set of α_i, β_i, γ_i, and δ_i suppose the corresponding values of the unknown quantities are

$$(7) \qquad x + \xi, \quad y + \eta, \quad z + \zeta.$$

It is supposed that the determinant is not zero with the coefficients (6).

With the notation that has been adopted the equations become

$$(8) \qquad \begin{cases} (a_1 + \alpha_1)(x + \xi) + (b_1 + \beta_1)(y + \eta) + (c_1 + \gamma_1)(z + \zeta) = d_1 + \delta_1, \\ (a_2 + \alpha_2)(x + \xi) + (b_2 + \beta_2)(y + \eta) + (c_2 + \gamma_2)(z + \zeta) = d_2 + \delta_2, \\ (a_3 + \alpha_3)(x + \xi) + (b_3 + \beta_3)(y + \eta) + (c_3 + \gamma_3)(z + \zeta) = d_3 + \delta_3. \end{cases}$$

After subtracting equations (4) the equations defining ξ, η, and ζ are

$$(9) \qquad \begin{cases} (a_1 + \alpha_1)\xi + (b_1 + \beta_1)\eta + (c_1 + \gamma_1)\zeta = \delta_1 - \alpha_1 x - \beta_1 y - \gamma_1 z, \\ (a_2 + \alpha_2)\xi + (b_2 + \beta_2)\eta + (c_2 + \gamma_2)\zeta = \delta_2 - \alpha_2 x - \beta_2 y - \gamma_2 z, \\ (a_3 + \alpha_3)\xi + (b_3 + \beta_3)\eta + (c_3 + \gamma_3)\zeta = \delta_3 - \alpha_3 x - \beta_3 y - \gamma_3 z. \end{cases}$$

The expression for ξ is

$$(10) \qquad \xi = \frac{\begin{vmatrix} \delta_1 - \alpha_1 x - \beta_1 y - \gamma_1 z, & b_1 + \beta_1, & c_1 + \gamma_1 \\ \delta_2 - \alpha_2 x - \beta_2 y - \gamma_2 z, & b_2 + \beta_2, & c_2 + \gamma_2 \\ \delta_3 - \alpha_3 x - \beta_3 y - \gamma_3 z, & b_3 + \beta_3, & c_3 + \gamma_3 \end{vmatrix}}{\begin{vmatrix} a_1 + \alpha_1, & b_1 + \beta_1, & c_1 + \gamma_1 \\ a_2 + \alpha_2, & b_2 + \beta_2, & c_2 + \gamma_2 \\ a_3 + \alpha_3, & b_3 + \beta_3, & c_3 + \gamma_3 \end{vmatrix}}.$$

The problem is to find the greatest numerical value of ξ for admissible values of the α_i, β_i, γ_i, and δ_i.

The denominator determinant must have the same sign for all admissible values of the α_i, β_i, and γ_i in order that the original problem shall be at all determinate, for otherwise ξ is not bounded either in the negative or the positive direction. Suppose the α_i, β_i, and γ_i are so small that terms of the second degree in these quantities are relatively insensible; then the denominator of (10) becomes

$$(11) \qquad D_1 = \begin{vmatrix} a_1, & b_1, & c_1 \\ a_2, & b_2, & c_2 \\ a_3, & b_3, & c_3 \end{vmatrix} + \begin{vmatrix} \alpha_1, & b_1, & c_1 \\ \alpha_2, & b_2, & c_2 \\ \alpha_3, & b_3, & c_3 \end{vmatrix} + \begin{vmatrix} a_1, & \beta_1, & c_1 \\ a_2, & \beta_2, & c_2 \\ a_3, & \beta_3, & c_3 \end{vmatrix} + \begin{vmatrix} a_1, & b_1, & \gamma_1 \\ a_2, & b_2, & \gamma_2 \\ a_3, & b_3, & \gamma_3 \end{vmatrix}.$$

The first of these determinants, which is D of equation (5), is small in case of the equations which arise in the determination of orbits because the three columns are approximately equal. The numerical example with which this discussion was started illustrates the matter. When this condition is not satisfied initially it can be brought about, in case D is small, by linear combinations of the rows of the determinant. Hence it will be supposed that this condition is satisfied, and further that the elements of the second column are greater than those of the first, and that those of the third column are greater than those of the second.

The expansion of the second determinant of (11) is

$$\alpha_1(b_2c_3 - b_3c_2) - \alpha_2(b_1c_3 - b_3c_1) + \alpha_3(b_1c_2 - b_2c_1).$$

Let the numerically largest of the three parentheses be represented d_2. Then the second determinant of (11) can not exceed $3\delta d_2$ in numerical value, where δ is the greatest numerical value the α_i can take. There are corresponding results for the third and fourth determinants of (11). Let d be the value of the numerically greatest of d_2, d_3, and d_4. Then the sum of the last three determinants of (11) can not exceed in numerical value $9\delta d$. The value of δ is five units in the first neglected place. Consequently 9δ is about five units in the last place. But because of the approximate equality of the columns of the determinant d is also a small quantity. In the given numerical example, which is comparable to what often arises in practice, d is less than .02. In this example $9\delta d$ is less than one unit in the sixth place, while D, the first determinant of (11), is more than fifty times as great. In all such cases where D has one or more significant figures which are distinct from zero, the first significant figure of ξ is independent of the α_i, β_i, and γ_i so far as they appear in the denominator of (10). When this condition is not satisfied the problem is practically indeterminate and further discussion is useless. When this condition is satisfied the difference of the second and third columns of the numerator of (10) will have at least one significant figure which is distinct from zero, and consequently in these columns the β_i and γ_i can be neglected. Therefore when D has one or more significant figures distinct from zero, at least the first significant figure of ξ is determined by

$$(12) \qquad D\xi = \begin{vmatrix} \delta_1 - \alpha_1 x - \beta_1 y - \gamma_1 z, & b_1, & c_1 \\ \delta_2 - \alpha_2 x - \beta_2 y - \gamma_2 z, & b_2, & c_2 \\ \delta_3 - \alpha_3 x - \beta_3 y - \gamma_3 z, & b_3, & c_3 \end{vmatrix}.$$

The numerical value of ξ is greatest when the α_i, β_i, γ_i, and δ_i have such signs that all the terms of the expansion of (12) have the same sign and their largest possible numerical values. Suppose the α_i, β_i, γ_i, and δ_i all have the same greatest

possible numerical value δ. Let $|x|, |b_2c_3 - b_3c_2|, \cdots$ represent the numerical values of $x, b_2c_3 - c_3c_2, \cdots$; then the greatest numerical value of ξ is

$$(13)\quad \xi = \frac{\delta}{|D|}\{1 + |x| + |y| + |z|\}\{|b_2c_3 - b_3c_2| + |b_3c_1 - b_1c_3| + |b_1c_2 - b_2c_1|\},$$

which is positive or negative according as δ is taken positive or negative.

There are expressions corresponding to (13) which give limits on the numerical values of η and ζ. They differ from (13) only in that b_i is replaced by a_i and c_i by a_i respectively. Since the signs of the α_i, β_i, γ_i, and δ_i were all determined in making ξ a maximum, in general η and ζ will not also have their greatest numerical values. Necessary and sufficient conditions that ξ, η, and ζ shall simultaneously have their maximum numerical values are that the ratios

$$(b_2c_3 - b_3c_2) : (b_3c_1 - b_1c_3) : (b_1c_2 - b_2c_1),$$

$$(a_2c_3 - a_3c_2) : (a_3c_1 - a_1c_3) : (a_1c_2 - a_2c_1),$$

$$(a_2b_3 - a_3b_2) : (a_3b_1 - a_1b_3) : (a_1b_2 - a_2b_1),$$

shall have the same signs in the three lines.

It is obviously improbable that in any physical problem the errors in the observations should all have the precise signs and numerical values to make ξ, η, or ζ a maximum numerically. There is a temptation to draw from this fact an erroneous conclusion, viz., that it is probable that ξ, η, and ζ should be near their mean values, or zero. This would be true if they were determined by a series of measurements whose numerical extremes are given by (13) and the two corresponding equations. But the hypothesis is that the observations give no presumption whatever as to the values of the α_i, β_i, and γ_i within the specified limits. This is actually the case in physical measurements. If many measurements are made the limits on the α_i, β_i, and γ_i are restricted, possibly, by taking their mean; but when they have once been determined any values within them are as improbable as, and no more improbable than, any other values. Consequently, there is no presumption in any particular case that ξ is near its mean value rather than near one of its extremes. In fact, equation (12) gives the true measure of uncertainty in the determination of ξ.

Equation (13) shows how the uncertainty in x depends upon the determinant D. Other things being equal the smaller D the greater the uncertainty in the solution. In the case of the numerical example considered above, equation (13) gives a total range in ξ of twelve units in the third decimal. The results given in (3) differ by eight units in the third decimal, showing that even this large variation in the solution is not the greatest possible. The value of y is uncertain nearly to two units in the second decimal, and z to twelve units in the third decimal.

An interesting point is illustrated by the numerical example which has been under discussion. The solution of (1), regarding the digits beyond the fifth place as being unknown, is determinate only to two decimal places; yet the equations will not *in general* be exactly satisfied if x, y, and z are varied in the fifth decimal. Since any digits in the fifth place in the solution are permissible, the fact that a solution does not exactly satisfy the equations does not prove that it is not correct so far as it is determinate; and the fact that it does exactly satisfy the equations does not prove that it is correct beyond where it ceases to be determinate. In general, when the determinant of a linear system of equations is small, the test of accuracy of the solution by direct substitution loses much of its value.

The determinant of the coefficients of a linear system of equations has a geometrical meaning which makes intuitionally clear the fact that when it is small the solution is poorly determined. For simplicity, consider the three equations (4). Each is the equation of a plane, and their common point of intersection is the solution of the system of equations. The direction cosines of the perpendiculars to these planes are proportional to the coefficients of x, y, and z. When the equations are multiplied by such factors that $a_i^2 + b_i^2 + c_i^2 = 1$, the direction cosines are a_i, b_i, and c_i. Suppose this has been done and consider the three perpendiculars which pass through the origin. They intersect the unit sphere whose center is the origin in points whose coördinates are the a_i, b_i, c_i $(i = 1, 2, 3)$. The determinant D is then, by solid geometry, six times the volume of the tetrahedron whose four vertices are the origin and the three points a_i, b_i, c_i. The volume of the tetrahedron is small if the three points are close together, or if they are nearly on the same great circle on the unit sphere. In the first cases the three planes are nearly parallel, when obviously small uncertainties in their precise positions correspond to large uncertainties in the point at which they intersect; and in the second case the line of intersection of any two of them is nearly parallel to the third, so that again their point of intersection is poorly determined.

There remains finally the question whether the fact that the solution is determinate to a smaller number of places than the original coefficients are given may not make it possible to shorten the computation by arranging the formulas so that tables with a small number of places can be used without loss of accuracy. If the solution is made by successive elimination of the unknowns the tables must contain at least as many places as there are significant figures in the coefficients, for significant figures are lost in the process by the addition of nearly equal quantities having opposite signs. But if the solution is made by determinants, the elements of one or more columns can be made small by simple linear combinations of the columns. Since the expansion of a determinant is made up of terms having a factor from each column, each term contains one or more small factors. In multiplying together several factors containing various numbers of significant figures, it is sufficient to use tables having one more place than appears in the smallest factor. Hence short tables can be used if one or more columns of the determinants have only small numbers.

The determinant solution of (1) for x is

$$x = \frac{\begin{vmatrix} .24561, & .35381, & .36518 \\ .62433, & .90274, & .91143 \\ .17145, & .23642, & .24375 \end{vmatrix}}{\begin{vmatrix} .34622, & .35381, & .36518 \\ .89318, & .90274, & .91143 \\ .22431, & .23642, & .24375 \end{vmatrix}}.$$

If in the numerator determinant two thirds of the third column is subtracted from the first column, and if the second column is subtracted from the third, the value of the determinant is not altered. In the denominator let the middle column be subtracted from each of the others, and then the third column added to the first. The resulting expression for x is

$$x = \frac{\begin{vmatrix} .00216, & .35381, & .01137 \\ .01671, & .90274, & .00869 \\ .00895, & .23642, & .00733 \end{vmatrix}}{\begin{vmatrix} +.00378, & .35381, & .01137 \\ -.00087, & .90274, & .00869 \\ -.00478, & .23642, & .00733 \end{vmatrix}}.$$

The computation can now be made with four-place tables containing all the numbers required on two pages. In computing x it is found in two or three minutes that to three figures its value is -1.03, which is as far as it is determinate.

THE UNIVERSITY OF CHICAGO,
May 19, 1913.

THE ACCURACY OF INTERPOLATION IN A FIVE-PLACE TABLE OF LOGARITHMS OF SINES.

By A. M. KENYON and G. JAMES, Purdue University.

In a table which gives the logarithms of sines for each minute of angle, the logarithms of sines of angles to tenths of minutes are obtained by interpolation.

The name *Ordinary Interpolation* will be used to denote the process based on the assumption:

(1) As the angle increases from m' to $(m+1)'$ the increment of the logarithm of its sine varies as the increment of the angle; that is,

$$\log \sin (m + n/10)' = \log \sin m' + (n/10)(\log \sin(m+1)' - \log \sin m')$$
$$(n = 1, 2, 3, \cdots 9).$$

The error introduced by ordinary interpolation arises from two sources:

(*a*) The *essential* error, due to the assumption (1).

(*b*) The *tabular* error, due to the fact that the tabulated values of $\log \sin m'$ and of $\log \sin(m+1)'$ are in error.

The essential error increases as the angle decreases and makes ordinary interpolation unreliable for small angles. The tabular error is irregular but nowhere in the table does it exceed, in absolute value, one unit in the fifth place.

To make this clear we need to note that whenever an irrational number is approximated by a rational number correct to n decimal places, an error is introduced which in absolute value is less than $5/10^{n+1}$. Thus if we write $\sin 45° = .70711$, the result is too large by an error between .000003 and .000004.

Let $\log \sin m' = a$, $\log \sin(m+1)' = b$ and let $a - e_1$, $b - e_2$, be the five place values of these same logarithms. Then by ordinary interpolation (assumed to be a correct process since we seek here only the tabular error), the interpolated value of $\log \sin(m + n/10)'$ would be

$$c = a + (n/10)(b - a) = c_1 - e_3$$

where c_1 represents the five place value of c.

But by ordinary interpolation from a five place table, the result will be

$$\begin{aligned} c_2 &= a - e_1 + (n/10)(b - e_2 - a + e_1) \\ &= a + (n/10)(b - a) - [e_1 + (n/10)(e_2 - e_1)] \\ &= c_1 - [e_3 + e_1 + (n/10)(e_2 - e_1)]. \end{aligned}$$

Whence the five place tabular error is

$$c_1 - c_2 = (\bar{n}/10)e_1 + (n/10)e_2 + e_3, \text{ where } n + \bar{n} = 10,$$

and obviously the absolute value of this error is less than .00001.

Actual trial of a very large number of cases (over 1,200) shows that ordinary interpolation below 2° gives the result correct to five places in less than 30 per cent. of all cases; that it gives an error of one in the fifth place in about 50 per cent., and an error of two or more in the fifth place in about 20 per cent. of cases. There is no case above 1° 30′ where the total error is greater than one unit in the fifth place. Above 2° it gives the result correct to five places in from 70 per cent. to 83 per cent. of all cases and in no case tried is the error greater than one unit in the fifth place.

The name *Logarithmic Interpolation* will be used to denote the process based on the following assumption:

(2) For sufficiently small angles, the sine varies as the angle; which gives, if m and n denote measures of angles in the same unit,

$$\log \sin(m + n/10) = \log \sin m + \log(m + n/10) - \log m.$$

Here the essential error decreases with the angle; the tabular error as before is irregular but nowhere in the table exceeds, in absolute value, one unit in the fifth place.

Actual trial of all cases (to tenths of minutes) shows that in no case below 5° 45′ does logarithmic interpolation introduce a total error greater than one in the

fifth place.[1] Below 5° it gives the result correct to five places in about 60 per cent. of all cases and shows an error of one in the fifth place in about 40 per cent. of cases.

Below 2°, logarithmic interpolation is certainly preferable to ordinary interpolation for three reasons: (1) it gives the correct result in a larger number of cases; (2) it gives an error in the fifth place in a smaller number of cases; (3) it never gives an error greater than one in the fifth place.

Above 2°, ordinary interpolation is preferable for the same three reasons. All cases between 1° 30′ and 2° 15′ were examined.

An examination of 200 cases in the neighborhood of 84° shows that ordinary interpolation will give the result correct to five places in about 83 per cent., and will show an error of one in the fifth place in 17 per cent. of cases.

Below is a table of results of the statistical investigation.

Interval of Angle.	Number of Cases Examined.	Per Cent. of Cases Giving Result Correct to 5 Places.		Per Cent. of Cases Showing an Error of One in the 5th Place.		Per Cent. of Cases Showing an Error of Two or More in the Fifth Place.	
		Ordinary.	Logarithmic.	Ordinary.	Logarithmic.	Ordinary.	Logarithmic. [2]
1′–1°	45	Fails	60	Large	40		0
1° –1° 30′	97	33		57		10	0
1° 30′–1° 45′	All	55	63	45	37	0	0
1° 45′–2°	All	63	69	37	31	0	0
2° –2° 15′	All	71	62	29	38	0	0
2° 15′–3°	55	78	60	22	40	0	0
3° –4°	55	80		20		0	0
4° –5°						0	0
5° –5° 45′	All		66		34	0	0
5° 45′–6°	All		64		36		[3]

A THEOREM ABOUT ISOGONAL CONJUGATES.

By DAVID F. BARROW, Harvard University.

As an introduction let us recall two well-known theorems of elementary plane geometry:

THEOREM I. *Given a triangle $A_1A_2A_3$ (Fig. 1) and any point P not a vertex, join P to the three vertices and reflect each of the three lines thus drawn in the bisector of the angle at the corresponding vertex. The three reflected lines will meet in a point P′, called the isogonal conjugate of P.*

THEOREM II. *Given a triangle and a point marked at random on each side. If three circles be drawn, one through each vertex and the two adjacent marked points, these three circles meet in a common point.*

[1] The first error of two in the fifth place enters at 5° 53′.7

[2] All cases examined for this column.

[3] At 5° 53′.7 the first error of 2 in the fifth place enters.

We propose to prove a new theorem:

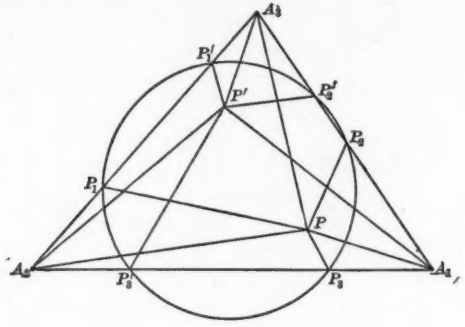

FIG. 1.

THEOREM III. *Given a triangle $A_1A_2A_3$ (Fig. 2) and a circle cutting each side in two points, P_1, P_1'; P_2, P_2'; P_3, P_3', where P_i and P_i' are on the side opposite A_i. If a point P be located as the common intersection of the three circles through the three sets of points $P_iA_jP_k$; and a point P' as the common intersection of three circles through $P_i'A_jP_k'$; then P and P' are isogonal conjugates.*

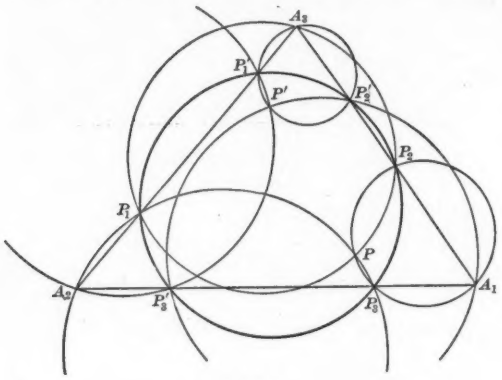

FIG. 2.

Lemma. Let P and P' be isogonal conjugates with regard to the triangle $A_1A_2A_3$ (Fig. 1).

Choose three points P_1, P_2, P_3, one on each side of the triangle, so that

$$\measuredangle PP_1A_2 = \measuredangle PP_2A_3 = \measuredangle PP_3A_1 = \theta$$

and choose three other points P_1', P_2', P_3', one on each side, and such that

$$\measuredangle P'P_1'A_3 = \measuredangle P'P_2'A_1 = \measuredangle P'P_3'A_2 = \theta$$

where θ is any angle. Then we can show that these six chosen points all lie on a circle.

From the similar triangles PA_1P_3 and $P'A_1P_2'$

$$\frac{A_1P_3}{A_1P} = \frac{A_1P_2'}{A_1P'}$$

and from similar triangles PA_1P_2 and $P'A_1P_3'$

$$\frac{A_1P_3'}{A_1P'} = \frac{A_1P_2}{A_1P}$$

Multiply these two equations together, canceling denominators.

$$A_1P_3 \times A_1P_3' = A_1P_2 \times A_1P_2'.$$

Hence P_2, P_2', P_3 and P_3' are four concyclic points. In like manner P_1, P_1', P_2, P_2', and also P_3, P_3', P_1, P_1' are concyclic. This gives us, apparently, three circles on each of which lie four of the six points. Now the sides of the triangle are the three radical axes of these circles taken two at a time. All three circles cannot be distinct because these three radical axes do not concur in a point. Hence at least two circles coincide, and so all six points are concyclic.

Observe also that by the construction of Fig. 1

$$\measuredangle PP_iA_j + \measuredangle PP_kA_j = 180°$$

and so P, P_i, A_j, P_k are concyclic. Therefore P is the point where the three circles $P_iA_jP_k$ meet. Similarly P' is the intersection of the three circles $P_i'A_jP_k'$.

We get the proof of Theorem III as follows:

Suppose we start with the triangle $A_1A_2A_3$; put down a circle cutting its sides in points P_1, P_1'; P_2, P_2'; P_3, P_3'; and locate P as the point common to the three circles $P_iA_jP_k$. Then the lemma shows that the isogonal conjugate of P lies on each of the three circles $P_i'A_jP_k'$; and since only one point does this, then the isogonal conjugate of P is that point.

If we interchange two of the points that lie an a side, say P_1 and P_1', we get a new pair of isogonal conjugates. Evidently four different pairs of isogonal conjugates can be obtained in this way, all determined by the same given circle cutting the sides. If each pair of isogonal conjugates be connected by a straight line and the perpendicular bisectors of these four lines be erected; then these four bisectors all meet in the center of the given circle. The only proof of this, which I have been able to devise, is rather tedious; but doubtless a simple and elegant proof exists.

If the given circle is tangent to one side of the triangle, but cuts the other two in distinct points, then it determines only two pairs of isogonal conjugates. If it be tangent to two sides it determines but one pair of isogonal conjugates. If it be an inscribed or escribed circle it determines one self-conjugate point, its center.

SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

By H. E. SLAUGHT, University of Chicago.

The twentieth summer meeting and the seventh colloquium of the American Mathematical Society were held at Madison, Wisconsin, during the week September 8–13, 1913. There were 57 members of the Society in attendance at the meetings, together with a considerable number of others who were directly or indirectly interested. The summer meeting of the Society is the most representative in attendance of any during the year. Professor H. F. Blichfeldt from Stanford University and Professor W. F. Osgood from Harvard University represented the far West and the far East; while Professor G. H. Ling from the University of Saskatchewan, Canada, and Professor Oscar Bolza from the University of Freiburg, Germany, added an international character to the gathering. The largest delegation was naturally from the University of Wisconsin on whose campus the meetings were held. Ten members of the Wisconsin staff were in attendance. The next largest delegation was from the University of Chicago with seven members of the faculty and three graduate students. Then followed the University of Minnesota with five members, the University of Illinois with four, Indiana University with three, Harvard University, Brown University, Northwestern University and the University of Michigan with two each, Stanford University, University of Iowa, University of Freiburg, University of Missouri, University of Saskatchewan, University of Cincinnati, Washington University, Dartmouth College, James Millikan University, Pennsylvania State College, Miami University, University of Oregon, and University of Wooster with one each. Important institutions in the East from which there was no representation were Yale, Columbia, Princeton, Cornell and Pennsylvania; and in the West, Kansas, Nebraska and California.

Conspicuous for their absence also were representatives of the many colleges, especially in the Middle West, where the distances were not insuperable. In this connection it may be said with some force that the usual hasty reading of brief outlines of highly abstract and technical papers has little interest for the average teacher of mathematics in the colleges, and hence the inducement is small for him to attend such meetings. It may well be that the Society has gone too far in confining its meetings, in general, to such a presentation of papers and that, even for the members who do attend, the more frequent presentation of expository papers would be welcomed. At any rate, those who attended the Colloquium were charmed with the clearness of exposition on the part of both speakers. Even those who had little knowledge of the fields covered were most agreeably surprised to find that they were able to understand practically all of the matter presented. This revelation and the further phenomenon of a similar nature with respect to a paper on the general program, by Professor Bôcher of Harvard University, on "The infinite regions of various geometries," which had been

scheduled for ten minutes but for which Professor Osgood, in presenting it, used forty minutes, to the great delight and satisfaction of all who heard him,—these experiences led many to declare on the spot that time should be given in all our programs for the presentation in detail of some expository papers.

The general program, including four half day sessions, contained 46 papers, several of which were read by title only, the authors in such cases not being present. The time allowance for the papers read averaged fifteen minutes each. It is impossible to present any details of a technical paper in this time. Those who attempt to do so usually fail even to make clear the salient points of their papers; but those who devote the time allotted to stating the historical setting, the nature of the problem which the paper is attacking, and the results reached, usually leave a clear impression on the minds of the auditors. On this occasion there was a larger number than usual who followed the latter method of presentation, but some still persisted in attempting the impossible. Even ten minutes used in telling what a paper is about and what conclusions are reached is better than twenty or twenty-five used in attempting to show details of proof for a paper which may have required a year in preparation.

But, after all, the most important feature of such meetings is the opportunity to meet the men who are doing things in mathematics. To spend a week in intimate social intercourse with the choice men of our profession is the best tonic for scientific activity which one can take. The Madison meetings were admirably arranged to provide a maximum of opportunity for this purpose. There was ample time in the mornings, at noontime, and in the afternoons and evenings for little gatherings at the University Club House, in the homes of the local members, in the libraries, and on excursions, to renew old acquaintances and to make new ones, to strengthen personal attachments and to gain inspiration from close touch with others. There were dinner groups at the homes of Professors Van Vleck, Mason and others; there was an automobile ride about the beautiful suburbs of Madison in machines owned by the local members and their friends; there was a tour of inspection about the campus and buildings of the University of Wisconsin; and there was an excursion on Lake Mendota in a specially chartered steamer which provided a two hours' social gathering of all the members, while the beautiful views of the Capitol, the University, and the highland borders of the lake formed a panoramic background. The boat ride ended at the Golf Club House where dinner was served and informal addresses were made, some in lighter vein, some in serious consideration of matters important to the Society. The one thing lacking was the presence of Professor F. N. Cole whose services as Secretary of the Society and Managing Editor of the *Bulletin*, for many years, have led us to think that a meeting can hardly be complete without him. In recognition of this universal sentiment a telegram was sent to him expressing regret for his enforced absence.

As intimated above, the colloquium was the great attraction of this meeting. There were five lectures by Professor W. F. Osgood, of Harvard University, on "Topics in the theory of functions of several complex variables," and five lectures

by Professor L. E. Dickson, of the University of Chicago, on "Certain aspects of a general theory of invariants, with special consideration of modular invariants and modular geometry." It was a great satisfaction to listen to two such masters of exposition as Professors Osgood and Dickson and to have the results of long investigation in important fields so clearly set forth that even non-specialists in those fields could follow and understand at least sufficiently for real enjoyment. These lectures will doubtless be published and will thus become available as important contributions in their respective fields.

Six other colloquia have been held by the Society at intervals of three or four years and thus a body of mathematical investigations is gradually forming, as one product of our own membership, which is unique in character and importance. The attendance of 51 auditors at this colloquium exceeded that of any previous one, the nearest approach being that of the Yale colloquium in 1907 when there was an attendance of 43. There were 25 at Ithaca in 1901, 31 at Boston in 1903, and 28 at Princeton in 1909. The Madison colloquium was also marked as the only one thus far held in the West. Indeed, it was the "farthest west" summer meeting except the one in St. Louis in connection with the World's Exposition in 1904.

BOOK REVIEWS.

W. H. BUSSEY, CHAIRMAN OF THE COMMITTEE.

College Algebra. By WILLIAM BENJAMIN FITE. D. C. Heath & Co., Boston, 1913. iv + 283 pages. $1.40.

The book covers the usual topics of elementary algebra in about the conventional order. In general, the treatment is lucid and the proofs are as rigorous as the limited comprehension of young students will permit. There are nineteen chapters on the following topics: The fundamental operations, factors and multiples, fractions, linear equations in one unknown, systems of linear equations in two or more unknowns, fractional and negative exponents and radicals, quadratics, systems of equations in two unknowns solvable by means of quadratics, progressions, permutations and combinations, mathematical induction, complex numbers, theory of equations, determinants, inequalities, partial fractions, logarithms, variation, and infinite series.

In Chapter I the fundamental operations are defined geometrically; the assumptions concerning real numbers are definitely stated, and the consequences drawn as theorems. While the teacher may find this treatment desirable, it will probably prove of little real value to those students for whom the book is intended.

In Chapter V one method of solution for systems of linear equations in two or more unknowns is by the use of second and third order determinants as formulae. Those teachers for whose purposes such an elementary treatment of

determinants is sufficient may omit Chapter XIV in which determinants are more extensively treated. In Chapter XI the treatment of mathematical induction is particularly commendable for the way in which it emphasizes both parts of the proof; namely, the direct verification of the theorem for the smallest admissible value of the integer, and the proof that if the theorem is true for one value of the integer, it is true for the next greater value.

In Chapter XII the concrete geometrical treatment of complex numbers is noteworthy. However, for students who have not studied trigonometry, articles 134–138, which involve the polar representation of complex numbers and their consequences, must be omitted.

In the last chapter the treatment of infinite series does not seem to belong to such an elementary text. It would seem preferable to omit it, and give a little broader discussion of "Limit," in Chapter IX, in treating infinite geometric series. The author's omission of the symbol for infinity and the word infinity itself, except as it is implied in the phrase "infinite series," in order to avoid the misinterpretation of ∞ as a definite, fixed number, seems very undesirable. The symbol is so convenient and is used so much in all mathematical literature, that the remedy against its misinterpretation is not to omit the symbol or word, but to retain it and give it the proper meaning.

The introduction of some of the principles and problems of analytic geometry is a desirable innovation in this text where graphical methods are used freely. In general, the book is by no means radical either in treatment or subject matter, and it will probably find favor with many teachers.

H. L. SLOBIN.

PROBLEMS AND QUESTIONS.

REMARKS BY THE MANAGING EDITOR.

The solutions of problems have not come in as promptly during the vacation season as at other times of the year. Hence none are given in this issue, those on hand being held over till November, when it is hoped that a full supply will be ready.

An examination of the problems proposed during this year, and of the solutions given, reveals the fact that, for the most part, solutions are not forthcoming until several months after the problems are proposed, showing that there is an *abiding interest* among the readers of the MONTHLY in these problems and that, even when solutions are not sent in immediately, they may still be expected later on.

Below is given an analysis of the situation so far as concerns the present year. We are glad to publish solutions of problems whenever they may have been proposed, but we are especially anxious to clear up by December, if possible, those thus far proposed this year, and we would urge all who are interested in this department to send in solutions to the Committee as promptly as may be.

In the ALGEBRA list, there have been proposed so far this year problems 383–395 inclusive, but solutions have been received for 383 and 384 only. The other solutions received and printed were for 376–380 inclusive, which were proposed last year.

In GEOMETRY, the problems proposed this year are 412–423 inclusive, and of these solutions have been received for 412, 413, 414 only. The other solutions published this year are 392, 407–409, and 411.

In CALCULUS, problems 334–344 inclusive have been proposed this year, but none of these have been solved so far—at least no solutions have been received. However, solutions for 326 and 328–331, proposed last year, have been received and published.

In MECHANICS, problems 271–279 inclusive have been proposed this year, and none of them have been solved. The solutions published are for 250, 263 and 265 which were proposed last year. Under this heading note especially that a mistake in numbering has crept in; namely, in the June issue the numbers should be 276 and 277 instead of 271 and 272, and in the September issue they should be 278 and 279 instead of 273 and 274.

In NUMBER THEORY, the problems proposed this year are 187–196 inclusive, and no solutions for any of these have been received, the only one being for 185 proposed last year. Here, again, there is a mistake in numbering in the June and September issues. These numbers should be 191–196 instead of 187–192. Readers will please make these corrections and refer to these problems accordingly.

There are, therefore, still to be solved eleven problems in Algebra, nine in Geometry, eleven in Calculus, nine in Mechanics, and seven in Number Theory, making a total of 47 problems proposed this year and not solved. In addition to these there are some still remaining from last year which may be considered by any one interested. It should be said that solutions are in hand and not yet printed for Algebra 386 and Geometry 415, 416. Manifestly, it would be impossible to print the solutions of all outstanding problems in the November and December issues, but we will give the largest possible space for this purpose and will be pleased with an overflow for this department which is "everybody's department."

In this connection, the Managing Editor may be permitted to ask for cooperation on some matters of importance to him in saving time and making the solutions of problems appear in the best possible form. Will contributors please exercise care in regard to the following points in the preparation of manuscripts:

(1) Please *state the problem* as well as give its number and the name of the proposer, so that the readers may have a complete presentation before them on the printed page.

(2) If any figures are involved, please draw them *accurately on a separate sheet* for the use of the engraver, putting a rough sketch in the manuscript for use of the editors.

(3) Please give attention to *paragraphing and spacing*, especially where matter is to be displayed on the printed page, so that the compositor may see at a glance what form is desired.

(4) Please be as *concise and brief* as is consistent with clearness, omitting the more obvious details but emphasizing the important steps.

(5) Please write *clearly and on one side only* of a sheet of paper and put *not more than one* solution on the same sheet.

In conclusion, attention is called to the statement made in the January issue under the heading "Problems and Questions," where the hope was expressed that in time this department might become useful in throwing light upon real difficulties arising in connection with actual work in which we are engaged. Some problems proposed are of this character and it is hoped that still more may be in this line. In order to encourage such questions, in cases where it is not convenient to formulate a problem, and where the questioner may not care to advertise his difficulty, the editors will provide a section of miscellaneous questions and answers (withholding the names of both questioners and answerers when so requested). Such questions might well refer, for example, to the selection of subject matter and method of presentation in courses in collegiate mathematics, and they might elicit replies in the form of a short symposium or they might even lead in some cases to extended articles. On the other hand, these questions might refer to difficulties of interpretation or to obscure steps in some demonstration or to anything whatsoever of mathematical import on which the questioner honestly seeks light. We may not be able to answer all such questions, but, at least, we will try to do so or to find some one who can. Communications for the department of "Miscellaneous Questions" may be sent directly to the Managing Editor, or preferably to Professor R. D. Carmichael, Indiana University, Bloomington, Ind., who will have this special department in charge. Of course all such communications should be signed but the names will be withheld from publication if so requested.

NOTES AND NEWS.

FLORIAN CAJORI, CHAIRMAN OF THE COMMITTEE.

The attention of all is called to the remarks of the Managing Editor under the heading "Problems and Questions" of this issue.

MR. F. E. CARR, who is instructor in mathematics in Oberlin College, will spend next year on leave of absence in graduate study at the University of Chicago.

PROFESSOR GORDON N. ARMSTRONG, of Ohio Wesleyan University, who last March passed successfully his examination for the degree of doctor of philosophy at the Technische Hochschule in Munich, spent the second semester traveling in Europe and in visiting one or two of the more important universities in Germany.

In the *Journal of the Washington Academy of Sciences* (Vol. III, June 19, 1913) C. E. VAN ORSTRAND publishes tables of the exponential functions, e^x, e^{-x}, for $x = 0.0$ to $x = 32.0$, to either 20 decimals or 20 (sometimes more) significant figures.

W. C. EELLS, professor in Whitworth College, Tacoma, contributes to the *Bibliotheca mathematica* for July, 1913, a short article on "Numerals in Indian Languages of North America." A fuller account of this research will appear in the November issue of the MONTHLY.

PROFESSOR R. R. SHUMWAY of the department of mathematics of the University of Minnesota has been appointed a member of the administrative board of the College of Science, Literature, and the Arts.

ASSISTANT PROFESSOR GUY H. ALBRIGHT, who was at Harvard University during the second semester of last year, resumes his work at Colorado College this fall.

It will be of interest to many readers of the MONTHLY to know that abstracts of fifteen of the thirty-six German reports to the International Commission on the Teaching of Mathematics may be found in the *Teachers College Record* of March, 1912 (published by the Teachers College, Columbia University, 30 cents per copy).

DR. E. L. DODD, of the University of Texas, contributes to the *Bulletin of the Mount Weather Observatory* (issued in April, 1913) an interesting discussion of the question: "Is the average of measurements the best approximation for the true value or normal value?"

The University of California Publications in Mathematics for June, 1912, contain an article by HENRY W. STAGER, "On Numbers which Contain No Factors of the Form $p(kp + 1)$."

An article, "The method of monodromie with applications to three parameter quartic equations" by Professor R. P. BAKER, of the State University of Iowa, appeared in the March number of the *Annals of Mathematics*.

The psychology of the errors committed in elementary mathematics is the subject of an article which has appeared in recent numbers of *School Science and Mathematics*. It is written by Principal THOMAS J. MCCORMACK of the La Salle-Peru High School of La Salle, Ill.

Five candidates for the Doctorate in mathematics were awarded the degree at the autumn convocation at the University of Chicago; namely, W. L. MISER, instructor-elect at the University of Minnesota; F. M. MORRISON, instructor on leave of absence from the University of Washington; W. WARREN, instructor at the University of Manitoba; E. J. MOULTON, just promoted to an assistant professorship at Northwestern University; and W. C. KRATHWOHL, associate professor-elect at Ripon College. MISS MILDRED L. SANDERSON, who received the degree in June, is instructor-elect at the University of Wisconsin.

Ginn and Company have recently published "The Teaching of Arithmetic" by PROFESSOR DAVID EUGENE SMITH of Columbia University. The work has been prepared with a view to the needs of teachers' reading circles and of those who are giving instruction or supervising the work in arithmetic in the elementary

schools. Another book of the same class which has appeared recently is DR. A. W. STAMPER'S "Text-book on the Teaching of Arithmetic," published by the American Book Company. Dr. Stamper is head of the department of mathematics at the State Normal School in Chico, California.

The *Bibliotheca mathematica* for July, 1913, contains two short articles by L. C. KARPINSKI, the first on "Recorde's Whetstone of Witte," the second on "The Oxford Mathematician, John Caswell (1655–1712)." To the biographical detail about Caswell we add a statement from Bishop Berkeley which apparently refers to a publication of Caswell not noted by Karpinski. Speaking of the cone and the archimedean cylinder circumscribed about a sphere, Berkeley[1] says: *Idem quod Tacquetus, etiam Cl. Wallisius in additionibus et emendationibus ad cap. LXXXI. algebrae suae, a D. Caswello ope Arithmetices Infinitorum demonstratum exhibet.*

HEINRICH WEBER, professor of mathematics in the University of Strassburg and author of works on algebra, died on May 17 from apoplexy. He was president of the third International Congress of Mathematicians, which was held at Heidelberg in 1904, and he wrote a large number of mathematical articles. He is probably best known to Americans by his algebra, consisting of three large volumes, and by the Encyklopädie der Elementar-Mathematik.

The EULER-KOMMISSION has issued a circular in which attention is called to the fact that the publication of Euler's Complete Works is more expensive than had been anticipated. It appears now that the deficit might reach two hundred thousand francs, and it is proposed to organize a society whose members are to make annual contributions of at least ten francs during the period of about fifteen years required for the publication. These members are to receive annually a brief report of the progress of the publication and also portraits of Euler.

In the *Tôhoku Mathematical Journal* for May, 1913, DR. G. B. HALSTED traces the history of the prismoidal formulas. Newton (*Methodus Differentialis*, 1711; further carried out by Cotes, in the works of Newton collected posthumously, 1722) showed how an area or volume could be evaluated from parallel cross-sections, and especially that from three cross-sections, following at the same distance apart, we get approximately the enclosed segment if we add the outer sections to four times the mid-section and multiply the sum by a sixth of the distance between the outer sections. Maclaurin (1742, *Fluxions*, No. 848) showed that Newton's special rule gives the content exactly when every section parallel to the base is a function of its distance from it of a degree not higher than the third, $f(x) = n_0 + n_1x + n_2x^2 + n_3x^3$. In Halsted's *Mensuration* (1881) the applicability of Newton's three-term prismoidal formula was treated without the calculus; the term prismatoid was introduced for a polyhedron with no summits other than the vertices of two parallel faces. In the present article Halsted derives a criterion for two-term prismoidal formulas.

[1] *The Works of George Berkeley*, Vol. III, Oxford, 1871, p. 54.

In beginning the re-publication in full of the Final Report of the National Committee of Fifteen on Geometry Syllabus, *The Irish Journal of Education* makes the following significant statement:

"To Professor Slaught, of the University of Chicago, Managing Editor of the *American Mathematical Monthly*, and Chairman of the National (U. S. A.) Committee on Geometry Syllabus, we are indebted for permission to re-publish in our JOURNAL the Final Report of the Committee. Very many copies of this Report have been called for and distributed in America—about 22,000 altogether, we think. In Ireland, we are not mistaken, we think, in saying that ours is the only body which has troubled itself with this important Report. A couple of Boards directly connected with education were unrepresented at the Congress—Is this a justification of the official assertion that there is money for everything but education? It is left to a body of hard-working, badly treated, and indifferently paid teachers, to do something to maintain educational progress, and make known the advances in other countries. Truly there is something strange in this spectacle."

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NUMBER SYSTEMS OF THE NORTH AMERICAN INDIANS.*

By W. C. EELLS, Whitworth College.

The linguistic diversity of the Indians inhabiting the North American continent is one of the most remarkable features of world ethnology.[1] The late director of the Bureau of American Ethnology[2] says: "In philology North America presents the richest field in the world, for here is found the greatest number of languages distributed among the greatest number of stocks." The Bureau recognizes almost three score distinct linguistic families having no lexical resemblance, no apparent unity of origin, no relation to European or Asiatic languages. These "families" are further subdivided linguistically into 750 "tribes" or languages.[3]

* Presented to the American Mathematical Society, San Francisco Section, Oct. 20, 1913.

[1] *Bibliographical Note.*—It is impossible to give a comprehensive bibliography of this subject in a small space. Most of the material is in hundreds of separate vocabularies, grammars, dictionaries and discussions of the various languages of the American Indians. Considerable may be found in the reports of the Bureau of American Ethnology, Washington, D. C. Pillings Bibliographies, published by this Bureau from 1887 to 1894, contain references to most of the literature up to the date of their publication for the nine most important linguistic families. In addition may be mentioned as especially important: Conant, L. L., *The Number Concept*, N. Y., 1896; Cushing, F. H., "Manual Concepts," in *Amer. Anthropologist*, 1892, p. 289; Eells, M., "Indians of Puget Sound: Measuring and Valuing," in *Amer. Antiquarian*, Vol. 10, p. 174; Dixon, R. B., and Kroeber, A. L., "Numeral Systems of the Languages of California," in *Amer. Anthropologist*, N.S., Vol. 9, p. 663; Powell, J. W., "Linguistic Families of America, North of Mexico," in Seventh Ann. Rpt., Bur. of Eth., 1885–86; Trumbull, J. H., "On Numerals in American Indian Languages," in *Trans. Amer. Philological Assn.*, 1874, p. 41. Full credit can scarcely be given for each statement made in this paper. The above mentioned sources have been used freely, but even more the numerous dictionaries and grammars mentioned at the beginning of this note. A bibliography of about 300 titles prepared by the author is on file in the library of the University of Chicago, the Newberry Library, Chicago, and the library of the University of Wisconsin.

[2] Major J. W. Powell: 1st Ann. Rpt. Bur. Eth., 1879–80, p. 78.

[3] For a list of these families and tribes see Powell: 7th Ann. Rpt. Bur. Eth., p. 1; Hodge: *Handbook of American Indians*, article "Linguistic Families." Not all these languages were entirely distinct, but most of them were. They are analogous to the French, Spanish, and Italian as different "languages" of the same "family." It is a conservative statement to say that about 500 distinct languages have been spoken on the American continent.